Math 131 B, Lecture 1
Real Analysis

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Let (M, d) be a metric space. We define a new metric d' on M by

$$d'(x, y) = \begin{cases} d(x, y) & \text{if } d(x, y) < 1 \\ 1 & \text{if } d(x, y) \geq 1 \end{cases}$$

(a) [5pts.] Prove that d' is a valid metric on M.

Solution: Clearly $d'(x, y) \geq 0$ with equality if and only if $x = y$ and $d'(x, y) = d'(y, x)$. However, we need to check the triangle inequality carefully. Let $x, y, z \in M$. There are several cases.

- If $d(x, z), d(x, y), d(y, z) < 1$, then $d'(x, z) = d(x, z) \leq d(x, y) + d(y, z) = d'(x, y) + d'(y, z)$.

- If $d(x, z) \geq 1$, but $d(x, y), d(y, z) < 1$, we have $d'(x, z) = 1 \leq d(x, z) \leq d(x, y) + d(y, z) = d'(x, y) + d'(y, z)$.

- If at least one of $d(x, y)$ or $d(y, z)$ is greater than or equal to one, then $d'(x, z) \leq 1 < d'(x, y) + d'(y, z)$.

(b) [5pts.] Show that if $S \subset M$, S is open in (M, d) if and only if S is open in (M, d').

Solution: If S is open in (M, d), every point $x \in S$ has a neighbourhood $B_d(x; r_x) \subset S$. By shrinking this ball if necessary, we can insist that $r_x < 1$. However, $B_{d'}(x; r_x) = B_d(x; r_x)$, so x also has a neighbourhood contained in S with respect to the metric d'. We conclude that S is open in (M, d'). The converse is identical.

Problem 2.

(a) [5pts.] What does it mean for a sequence $\{x_n\}$ to converge in (M, d)?

Solution: We say that $\{x_n\}$ converges to some p in (M, d) if for every $\epsilon > 0$, there is some N such that $n \geq N$ implies that $d(x_n, p) < \epsilon$.

(b) [5pts.] Let $S \subset M$ and $p \in \overline{S}$. Prove that there is a sequence of points in S converging to p.

Solution: If $p \in S$, it suffices to take the constant sequence $x_n = p$. Otherwise, if $p \notin S$, then p is a limit point of S, so for all $n \in \mathbb{N}$, there is a point of S in $B(p; \frac{1}{n}) - \{p\}$. Choose such a point and call it x_n. Then $x_n \to p$.

Problem 3.

(a) [5pts.] Give a definition of a compact set.

Solution: We say that $S \subset M$ is compact if for every open covering $\mathcal{F} = \{A_\alpha\}$ of S there is a finite subcover $A_1, \cdots, A_n$ such that $S \subset \bigcup_{i=1}^n A_i$.

(b) [5pts.] Let S and T be subsets of a metric space M such that S is compact and T is closed in M. Prove that $S \cap T$ is compact.

Solution: Since S and T are closed in M, $S \cap T$ is closed in M, hence also closed in S. But closed subsets of compact sets are compact, so we are done.

Problem 4.

(a) [5pts.] Give a definition of a complete metric space.

Solution: We say (M, d) is a complete metric space if every Cauchy sequence $\{(x_n)\}$ in M converges in M.

(b) [5pts.] Which of the following metric spaces are complete? Briefly justify your answers.

- $S_1 = (0, 1] \times [2, 4)$ with the metric inherited from $\mathbb{R}^2$.
- S_2 a discrete metric space.
- $S_3 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ with the metric inherited from $\mathbb{R}^3$.

Solution: S_1 is not complete: consider the Cauchy sequence $\{(\frac{1}{n}, 2)\}$, which fails to converge in S. S_2 is complete because any Cauchy sequence in a discrete metric space is eventually constant, hence converges. S_3 is closed and bounded in $\mathbb{R}^3$, hence compact, hence complete.

Problem 5.

(a) [5pts.] State the Lindelöf Covering Theorem.

Solution: Let $S \subset \mathbb{R}^n$, and $\mathcal{F} = \{A_\alpha\}$ be any open cover of S. Then a countable subcover $A_1, A_2, \cdots$ of $\{A_\alpha\}$ also covers S.

(b) [5pts.] Let S be a set in $\mathbb{R}^n$ with the property that for every $\mathbf{x}$ in S, there is a ball $B(\mathbf{x}; r_{\mathbf{x}})$ such that $B(\mathbf{x}; r_{\mathbf{x}}) \cap S$ is countable. Prove that S is countable.

Solution: The sets $B(\mathbf{x}; r_{\mathbf{x}})$ cover S, so by the Lindelof covering theorem we can pick a countable subcollection $U_i = B(\mathbf{x}_i; r_{\mathbf{x}_i})$ for $i = 1, 2, 3, \cdots$ that covers $\mathbb{S}$. Since there are countably many points of S in each U_i, we just need to argue that the countable union of countable sets is countable, which we can do by making a grid and taking diagonals in the usual way.